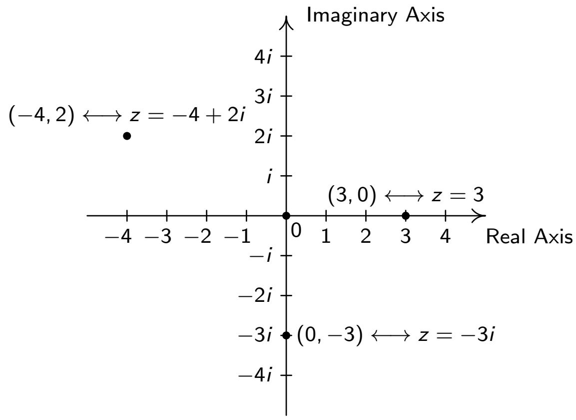
MATH 1700: SECTION 13.3: THE POLAR FORM OF COMPLEX NUMBERS

RECALL: A **complex number** is a number of the form $z = a + bi$ where a and b are real numbers and i is the imaginary unit defined by $i = \sqrt{-1}$. The number a is called the **real part** of z, denoted $\mathrm{Re}(z)$, while the real number b is called the **imaginary part** of z, denoted $\mathrm{Im}(z)$.

To start off this section, we associate each complex number $z = a + bi$ with the point (a, b) on the Cartesian (rectangular) coordinate plane. In this case, the x-axis is relabeled as the **real axis**, which corresponds to the real number line as usual, and the y-axis is relabeled as the **imaginary axis**, which is demarcated in increments of the imaginary unit i. The plane determined by these two axes is called the **complex plane**.



The Complex (Argand) Plane

Since the ordered pair (a, b) gives the *rectangular* coordinates associated with $z = a+bi$, the expression $z = a+bi$ is called the **rectangular form** of the complex number z.

We could just as easily associate z with a pair of *polar* coordinates (r, θ). Although it is not as straightforward as the definitions of $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, we give r and θ special names in relation to z below.

THE MODULUS AND ARGUMENT OF COMPLEX NUMBERS:

Let $z = a + bi$ be a complex number with $a = \mathrm{Re}(z)$ and $b = \mathrm{Im}(z)$. Let (r, θ) be a polar representation of the point with rectangular coordinates (a, b) where $r \geq 0$.

- The **modulus** of z, denoted $|z|$, is defined by $|z| = r$.
- The angle θ is an **argument** of z. The set of all arguments of z is denoted $\arg(z)$.
- If $z \neq 0$ and $-\pi < \theta \leq \pi$, then θ is the **principal argument** of z, written $\theta = \mathrm{Arg}(z)$.

Since every point in the plane has infinitely many polar coordinate representations (r, θ), it's worth our time to make sure the quantities 'modulus', 'argument' and 'principal argument' are well-defined. Concerning the modulus, if $z = 0$ then the point associated with z is the origin. In this case, the *only* r-value which can be used here is $r = 0$. Hence for $z = 0$, $|z| = 0$ is well-defined. If $z \neq 0$, then the point associated with z is not the origin, and there are *two* possibilities for r: one positive and one negative. However, we stipulated $r \geq 0$ in our definition so this pins down the value of $|z|$ to one and only one number.

Even with the requirement $r \geq 0$, there are infinitely many angles θ which can be used in a polar representation of a point (r, θ). If $z \neq 0$ then the point in question is not the origin, so all of these angles θ are coterminal. Since coterminal angles are exactly 2π radians apart, we are guaranteed that only one of them lies in the interval $(-\pi, \pi]$, and this angle is what we call the principal argument of z, $\text{Arg}(z)$.

The set $\arg(z)$ of all arguments of z can be described as $\arg(z) = \{\text{Arg}(z) + 2\pi k \mid k \text{ is an integer}\}$. Note that since $\arg(z)$ is a *set*, we will write '$\theta \in \arg(z)$' to mean 'θ is in the set of arguments of z'.

Note if $z = 0$ then the point in question is the origin, which we know can be represented in polar coordinates as $(0, \theta)$ for *any* angle θ. In this case, we have $\arg(0) = (-\infty, \infty)$ and since there is no one value of θ which lies $(-\pi, \pi]$, we leave $\text{Arg}(0)$ undefined.

EXAMPLE 1: Plot the following numbers in the complex plane. Find $\text{Re}(z)$, $\text{Im}(z)$, $|z|$, $\arg(z)$ and $\text{Arg}(z)$.

1. $z = \sqrt{3} - i$

2. $z = -2 + 4i$

3. $z = 3i$

4. $z = -117$

PROPERTIES OF THE MODULUS: Let z and w be complex numbers.

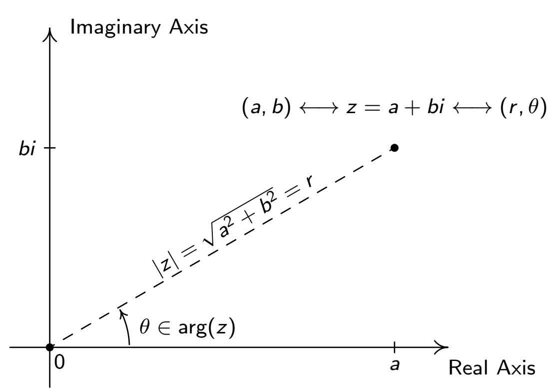
- $|z|$ is the distance from z to 0 in the complex plane
- $|z| \geq 0$ and $|z| = 0$ if and only if $z = 0$
- $|z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$
- **PRODUCT RULE:** $|zw| = |z||w|$
- **POWER RULE:** $|z^n| = |z|^n$ for all natural numbers, n
- **QUOTIENT RULE:** $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$, provided $w \neq 0$

PROPERTIES OF THE ARGUMENT: Let z be a complex number.

- If $\mathrm{Re}(z) \neq 0$ and $\theta \in \arg(z)$, then $\tan(\theta) = \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}$.
- If $\mathrm{Re}(z) = 0$ and $\mathrm{Im}(z) > 0$, then $\arg(z) = \left\{\frac{\pi}{2} + 2\pi k \mid k \text{ is an integer}\right\}$.
- If $\mathrm{Re}(z) = 0$ and $\mathrm{Im}(z) < 0$, then $\arg(z) = \left\{-\frac{\pi}{2} + 2\pi k \mid k \text{ is an integer}\right\}$.
- If $\mathrm{Re}(z) = \mathrm{Im}(z) = 0$, then $z = 0$ and $\arg(z) = (-\infty, \infty)$.

Proving these results amounts to working through the definitions and can be found in the text.

Our next goal is to completely marry the Geometry and the Algebra of the complex numbers:



Polar coordinates, (r, θ) associated with $z = a + bi$ with $r \geq 0$.

We know $a = r\cos(\theta)$ and $b = r\sin(\theta)$, hence: $z = a + bi = r\cos(\theta) + r\sin(\theta)i = r\left[\cos(\theta) + i\sin(\theta)\right]$.

The expression '$\cos(\theta) + i\sin(\theta)$' is abbreviated $\mathrm{cis}(\theta)$ so we can write $z = r\mathrm{cis}(\theta) = |z|\mathrm{cis}(\theta)$.

POLAR FORM OF A COMPLEX NUMBER:

If z is a complex number and $\theta \in \arg(z)$, $|z|\text{cis}(\theta) = |z|\left[\cos(\theta) + i\sin(\theta)\right]$ is called a **polar form** for z.

EXAMPLE 2:

1. Find the rectangular form of the following complex numbers. Find $\text{Re}(z)$ and $\text{Im}(z)$.

   (a) $z = 4\text{cis}\left(\frac{2\pi}{3}\right)$

   (b) $z = 2\text{cis}\left(-\frac{3\pi}{4}\right)$

   (c) $z = 3\text{cis}(0)$

   (d) $z = \text{cis}\left(\frac{\pi}{2}\right)$

2. Find a polar form of the following complex numbers.

   (a) $z = \sqrt{3} - i$

   (b) $z = -2 + 4i$

   (c) $z = 3i$

   (d) $z = -117$

PRODUCTS, POWERS, AND QUOTIENTS (OH MY!) IN POLAR FORM:

Suppose z and w are complex numbers with polar forms $z = |z|\text{cis}(\alpha)$ and $w = |w|\text{cis}(\beta)$. Then:

- **PRODUCT RULE:** $zw = |z||w|\text{cis}(\alpha + \beta)$
- **POWER RULE (DEMOIVRE'S THEOREM):** $z^n = |z|^n\text{cis}(n\theta)$ for every natural number n
- **QUOTIENT RULE:** $\dfrac{z}{w} = \dfrac{|z|}{|w|}\text{cis}(\alpha - \beta)$, provided $|w| \neq 0$

The proof of the above requires a healthy mix of definitions, arithmetic and identities and is found in the text.

EXAMPLE 3: Let $z = 2\sqrt{3} + 2i$ and $w = -1 + i\sqrt{3}$.

Find the following by converting z and w to polar forms. Write your answers in rectangular form.

1. zw

2. w^5

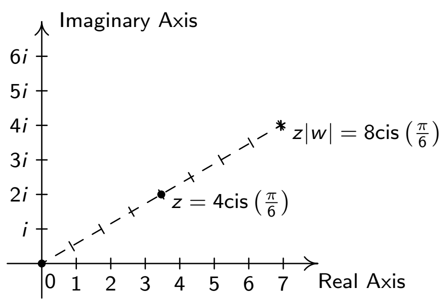
3. $\dfrac{z}{w}$

THE GEOMETRY OF ARITHMETIC IN POLAR FORMS:

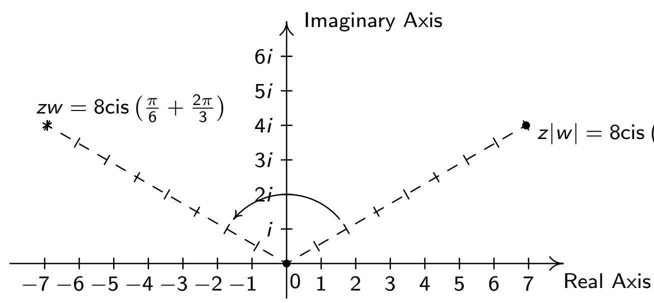
Take the product rule, for instance. If $z = |z|\text{cis}(\alpha)$ and $w = |w|\text{cis}(\beta)$, the formula $zw = |z||w|\text{cis}(\alpha + \beta)$ can be viewed geometrically as a two step process.

The multiplication of $|z|$ by $|w|$ can be interpreted as magnifying the distance $|z|$ from z to 0, by the factor $|w|$. Adding the argument of w to the argument of z can be interpreted geometrically as a rotation of β radians counter-clockwise.

Focusing on z and w from Example 3, we can arrive at the product zw by plotting z, doubling its distance from 0 (since $|w| = 2$), and rotating $\frac{2\pi}{3}$ radians counter-clockwise. The sequence of diagrams below attempts to describe this process geometrically.



Multiplying z by $|w| = 2$.

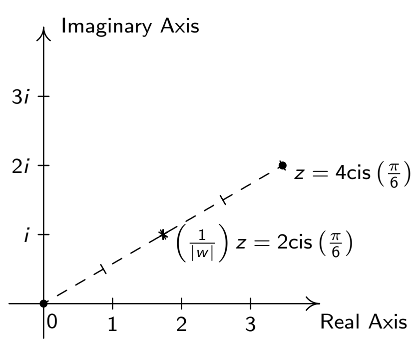


Rotating counter-clockwise by $\text{Arg}(w) = \frac{2\pi}{3}$ radians.

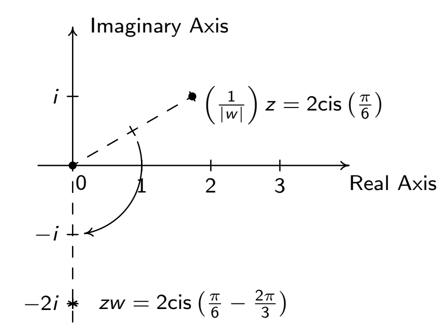
Visualizing zw for $z = 4\text{cis}\left(\frac{\pi}{6}\right)$ and $w = 2\text{cis}\left(\frac{2\pi}{3}\right)$.

We may also visualize division similarly. Here, the formula $\frac{z}{w} = \frac{|z|}{|w|}\text{cis}(\alpha - \beta)$ may be interpreted as shrinking the distance from 0 to z by the factor $|w|$, followed up by a *clockwise* rotation of β radians.

In the case of z and w from Example 3, we arrive at $\frac{z}{w}$ by first halving the distance from 0 to z, then rotating clockwise $\frac{2\pi}{3}$ radians as shown below.



Dividing z by $|w| = 2$.



Rotating clockwise by $\text{Arg}(w) = \frac{2\pi}{3}$ radians.

Visualizing $\dfrac{z}{w}$ for $z = 4\text{cis}\left(\frac{\pi}{6}\right)$ and $w = 2\text{cis}\left(\frac{2\pi}{3}\right)$.

ROOTS OF COMPLEX NUMBERS:

Let z and w be complex numbers. If there is a natural number n such that $w^n = z$, then w is an **n$^{\text{th}}$ root** of z.

THE n$^{\text{th}}$ ROOTS OF COMPLEX NUMBERS:

Let $z \neq 0$ be a complex number with polar form $z = r\text{cis}(\theta)$. For each natural number n, z has n distinct n^{th} roots, which we denote by w_0, w_1, ..., w_{n-1}, and they are given by the formula

$$w_k = \sqrt[n]{r}\text{cis}\left(\frac{\theta}{n} + \frac{2\pi}{n}k\right)$$

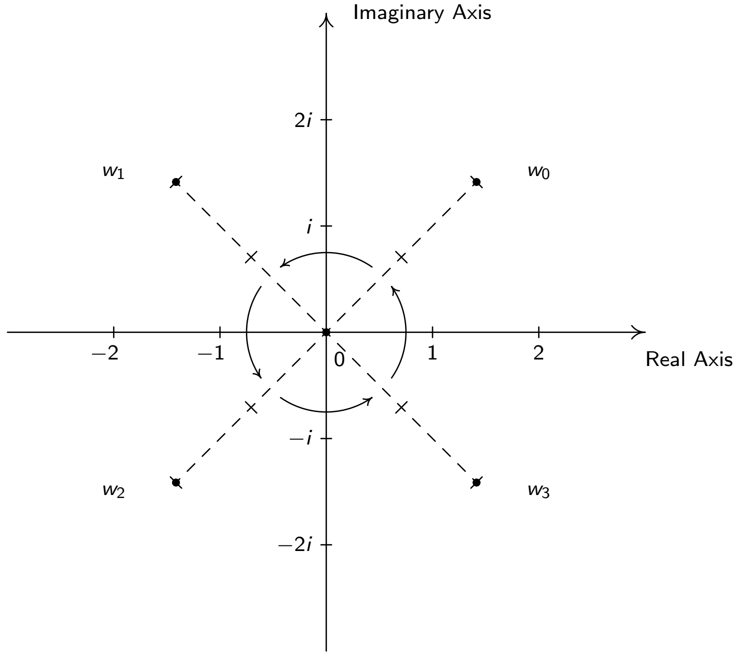
EXAMPLE 4: Find the following:

1. both square roots of $z = -2 + 2i\sqrt{3}$

2. the four fourth roots of $z = -16$

3. the six sixth roots of $z = 1$.

To find the n^{th} roots of a complex number, we first take the n^{th} root of the modulus and divide the argument by n. This gives the first root w_0. Each successive root is found by adding $\frac{2\pi}{n}$ to the argument, which amounts to rotating w_0 by $\frac{2\pi}{n}$ radians. The result is finding n roots, spaced equally around the complex plane. As an example of this, we plot our answers to number the four fourth roots of $z = -16$ below:



The four fourth roots of $z = -16$ equally spaced $\frac{2\pi}{4} = \frac{\pi}{2}$ around the plane.

HOMEWORK: Section 13.3: Page 1157: 1 - 73 every other odd; 84*